MA431: Homework 3

Required Reading

- Gelfand & Fomin, sections 3.13-3.14

Due April 4th at the start of class.

1. Gelfand & Fomin: 3.2. (Hint: Write $\Delta J = J(y+h) - J(y)$, then reuse everything we did in class, but include a few extra terms to account for the variation in G.)

2. Gelfand & Fomin: 3.4.

3. Gelfand & Fomin: 3.5. Include plots of the extremals. (Hint: First show that the general solution to both parts has the form $y^2 + (x-a)^2 = a^2$.)

4. Gelfand & Fomin: Problem 3.6. (Hint: See the solutions of Brachistochrone problem on pg. 26. Shift the solution slightly to match your problem.). Include plots of the extremals. To be specific, let the circles be $x^2 + y^2 = 1$ and $(x-a)^2 + (y-b)^2 = r^2$ with (a, b) in the first quadrant. Obviously the problem is not meaningful if the circles touch, so assume $\sqrt{a^2 + b^2} > 1 + r$.

5. Gelfand & Fomin: Return to problem 1.15bd. Set the interval to be $[0, 1]$ suppose $y(1) = a$ is fixed but that $y(0)$ is unknown. Find the extremals.

6. Suppose an external force is applied to move a particle from $x = 0$ to $x = 1$ through a force field $F = \langle u(x, y), v(x, y)\rangle$ with the goal of minimizing the work done. Assume the extremal path can be written in the form $y(x)$. Only account for work done to move the particle against the field and not any work done to accelerate the particle.

   (a) Construct the variational problem for this scenario.

   (b) State the corresponding Euler-Lagrange equation and boundary conditions.

   (c) Confirm that the field $F = \langle -y^2, 0\rangle$ has an extremum path which is obvious in hindsight. Is it a minimizer or maximizer?

   (d) Confirm that the inverse square field $F = \frac{\langle x-2, y\rangle}{((x-2)^2+y^2)^{3/2}}$ created by placing a point charge at $(2, 0)$ has an extremal path which is obvious in hindsight. Is it a minimizer or maximizer?

   (e) The force field is said to be conservative if it is the gradient of a potential function, that is $F = \nabla\phi$. Discuss the Euler-Lagrange equation for a conservative field.

   (f) The force field is said to be curl free if $\nabla \times F = 0$. How would your result from part b) be effected if F were curl free?

   (g) The force field is said to be divergence free if $\nabla \cdot F = 0$. Show that, if F is divergence free then $\Delta F = 0$ everywhere along an extremal, provided that second partials exist.

Additional Practice

MacCleur: 4.36, 5.35, and revisit the problems CVP1-10 in chapter 3. Write boundary conditions for cases where the endpoints may be unknown. Work out cases where either x, y, or both are unknown at one of the ends.

Gelfand & Fomin: Revisit problem 1.15. Assume the start point is $(0, 0)$ and that the end point is on the curve $f(x, y) = 0$.