MA212: Extra Assignment

Any problems marked with * require the use of maple. All other problems are to be done by hand.

1. Suppose a water purification system is composed of three 1000 liter holding ponds, each of which pumps liquid to and from the other two.

   - Fluid from pond 1 is being pumped into pond 2 at rate 1 L/s and into pond 3 at rate 2 L/s.
   - Fluid from pond 2 is being pumped into pond 1 at rate 2 L/s and into pond 3 at rate 1 L/s.
   - Fluid from pond 3 is being pumped into pond 1 at rate 1 L/s and into pond 2 at rate 2 L/s

   Ponds 2 and 3 initially contain pure water. Pond 1 initially has a contaminant concentration of 5g/L.

   (a) Setup a matrix ODE initial value problem describing this system.

   (b) * Solve the problem you wrote in part a).

   (c) Does the contaminant level in the ponds oscillate?

   (d) What happens to the contaminant level in the long term?

2. Radioactive ^{210}Pb undergoes beta decay to become ^{210}Bi which again undergoes beta decay to become ^{210}Po. These three all undergo alpha decay to become ^{206}Hg, ^{206}Tl, and ^{206}Pb, respectively. ^{206}Pb is stable and doesn't undergo any additional decay. Decay proceeds at a rate proportional to the amount present, with the rate constants for these decays given below. Ignore all other possible decays and assume that the only element present initially is 1 gram of unstable ^{210}Pb.

$$\begin{aligned}
&{}^{210}\mathrm{Pb} \longrightarrow {}^{210}\mathrm{Bi}, \quad 322 \text{ day}^{-1}\\
&{}^{210}\mathrm{Bi} \longrightarrow {}^{210}\mathrm{Po}, \quad 0.138 \text{ day}^{-1}\\
&{}^{210}\mathrm{Pb} \longrightarrow {}^{206}\mathrm{Hg}, \quad 8.55 \times 10^{-5} \text{ day}^{-1}\\
&{}^{210}\mathrm{Bi} \longrightarrow {}^{206}\mathrm{Tl}, \quad 0.138 \text{ day}^{-1}\\
&{}^{210}\mathrm{Po} \longrightarrow {}^{206}\mathrm{Pb}, \quad 1.37 \times 10^{-5} \text{ day}^{-1}
\end{aligned}$$

   (a) Write a matrix ODE initial value problem for the amounts of each element.

   (b) * Solve the problem you wrote in part a).

   (c) How much time will be required for half of the initial unstable Pb to be converted into stable Pb?

3. Consider the ODE system

$$
\begin{aligned}
x'' - y + x &= 1 \\
y'' + y' + x' - y &= 0
\end{aligned}
$$

   (a) Convert this to a system of four first order ODE.
   (b) * Find the general solution.
   (c) Write the expression for $y(t)$.

4. Two identical warm objects are placed into a cold room. The objects are touching each other. The objects give off heat, causing the air in the room to become warmer.

$$
\begin{aligned}
x' &= k(z - x) + q(y - x) \\
y' &= k(z - y) + q(x - y) \\
z' &= -q(z - x) - q(z - y)
\end{aligned}
$$

   (a) Explain why the model above describes this situation. Be explicit about the meaning of all variables and parameters.
   (b) Write this as a matrix ODE system and compute the general solution.
   (c) Are there critical points? If so, what are their stability properties?
   (d) Based on your answer to part b), what do you think happens in the long term? Support your answer with intuition about the physical problem.