Laplace Transforms

Problems marked ∗∗ are difficult and should not be attempted until you've solved the others.

1. Compute the Laplace Transform of each function using the definition. For problems marked with ∘, Maple can be used to compute integrals.

   (a) ∘ $\ln(t)$

   (b) $\sinh t$

   (c) $f(t) = \begin{cases} h & a < t < b \\ 0 & \text{otherwise} \end{cases}$, with $\{a, b, h\}$ constants.

   (d) $\delta(t-2)e^{-t^2}U(t-1)$ (Definition: $\int_0^\infty \delta(t-a)f(t)dt = f(a)$ for any $a > 0$.)

   (e) $te^{2t}\cos(\omega t)$

   (f) ∘ $\sqrt{t}$

2. Compute the Laplace Transform of each function using the table.

   (a) $\cosh t$

   (b) $(t+1)^2e^{-3t}$

   (c) $\sin(5-t)$

   (d) $t^2 \sin 2t$

   (e) $U(t-1)(t+1)^3$

   (f) $U(t-2)te^{-3t}$

   (g) A periodic square wave with period 2.

   (h) ∗∗ $\frac{\sin t}{t}$

   (i) $\int_0^t x\sin(t-x)dx$

   (j) ∗∗ $\cos^3(t)$

3. Compute the Inverse Laplace Transform of each function using the table.

   (a) $\frac{s}{(s-1)^4}$

   (b) ∗∗ $\frac{s}{(s-1)^4}$ (w/o partial fractions)

   (c) $\frac{s}{(s^2+1)^2}$

   (d) $\frac{2s-1}{s^2+2s+3}$

   (e) $\frac{e^{-s}}{1-s}$

   (f) $\frac{1}{(s^2-1)^2}$

   (g) ∗∗ $\frac{1}{(s^2-1)^2}$ (w/o partial fractions)

   (h) ∗∗ $\frac{e^{-3s}}{1-e^{-5s}}$ (There are at least two methods.)

4. Solve the following initial value problems.

   (a) $y''(t) + 4y(t) = -3\sin 2t,\ y(0) = 0,\ y'(0) = 0$

   (b) $y''' - 8y = 0,\ y(0) = 1,\ y'(0) = 0,\ y''(0) = 0$

   (c) $y'' - 4y' + 4y = (t-1)^2,\ y(0) = 1,\ y'(0) = 2$

   (d) $2y'' - 2y' + y = U(t-\pi)\sin t + 3\delta(t-2), y(0) = 0, y'(0) = 1$

   (e) ∗∗ $tx''(t) + 4x'(t) + 4tx(t) = 0, y(0) = 1, y'(0) = 0$

5. To make a ferrofluid, a 1kg box of powdered iron is poured into a bucket of oil at a rate of 5g/s. The bucket initially contains 10L of pure oil. After 10s, someone bumps the person pouring, causing 0.1kg of powder to go into the bucket all at once. The person pouring pauses for 5s to call the clumsy person an idiot. Pouring is resumed until the box is empty. Assume that adding this amount of powder to this volume of oil changes the volume a negligible amount. (This is almost correct.) Once the box is empty, additional oil is poured in at a rate of 10ml/s.

   (a) Create an ODE model describing this situation.

   (b) Use Laplace transforms to find the concentration of iron powder in the mixture and plot it versus time.

   (c) Behold the awesomeness of ferrofluids: tinyurl.com/ll5jm2w